Bounds on the length of magnetic field lines in a two-dimensional plasma

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Magnetic field lines in ideal turbulent plasmas tend to become quite complicated and their length to grow in time. Diffusivity allows for reconnection and possible shortening, but this fact has not so far been rigorously quantified. We show that in a two-dimensional diffusive plasma the mean length of field lines stays bounded for all time. Moreover, these estimates are local, in the sense that the mean values of magnetic field and velocity in the neighborhood of a ball determine bounds for length within the ball, without recourse to external magnitudes.

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I. INTRODUCTION

Magnetic field lines in ideal plasmas are transported by the flow as material points. As a result, if the fluid flow is chaotic or simply very complex, the configuration of field lines tends to become extremely complicated in time and their length to increase markedly. The presence of a positive resistivity has a smoothing effect upon the field gradient, but that does not mean *a priori* that field lines must become simpler. It is true, however, that now field lines may reconnect and eventually simplify the geometry, releasing in the process large amounts of magnetic energy, a fact that is believed to be central to the explanation of solar and stellar eruptive phenomena (see, e.g., [1,2] and references therein). We will show that, at least in dimension 2, resistivity indeed provides a bound for the spatial and temporal mean length of magnetic field lines: while some of them may at some time be very convoluted, on the whole they are not. Moreover, the sizes of magnetic and kinetic energy within a region provide a similar bound for this region, which proves that the shortening effect of resistivity is local: this is not obvious *a priori*, because some tangled configuration could be transported somewhere else by the flow, increasing mean length there. It must be noted that the means we will take are somewhat weighted by the size of $|\mathbf{B}|$: identical configurations of field lines have a larger mean if $|\mathbf{B}|$ is larger. The immediate reason why we cannot extend these results to dimension 3 is mathematical: only in a plane does there exist a scalar vector potential satisfying an induction equation amenable to analysis. If there is a physical obstruction is not clear so far. Indeed, the behavior of vector fields is very different in dimensions 2 and 3: the latter may become entangled, which is related to the capacity to store energy [3], and with the fact that two-dimensional dynamos do not exist [4,5]. It is also intuitively obvious that reconnection is easier in a plane. Still, the effect of diffusion is analogous in any dimension, so it is likely that some bounds exist for the three-dimensional case. In dimension 2, however, we may refine some techniques created to study passive scalar equations [6] in order to achieve our end.

Let us consider the mathematical setting of the problem. Assume that an incompressible plasma with resistivity η and fluid velocity $\mathbf{u}$ satisfies the induction equation

$$\partial \mathbf{B}/\partial t = \eta \Delta \mathbf{B} + \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{1}$$

within a simply connected domain Ω of the plane whose boundary is a piecewise differentiable curve. Incompressibility means that $\nabla \cdot \mathbf{u} = 0$, and, since we may assume that the fluid does not cross the boundary, $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$. Also, $\nabla \cdot \mathbf{B} = 0$, and $\mathbf{B}$ is assumed known at the initial time $t = 0$.

The induction equation is part of the full magnetohydrodynamics (MHD) system: the velocity satisfies the momentum equation with a forcing term equal to the Lorentz force $(\nabla \times \mathbf{B}) \times \mathbf{B}$, plus possibly other material forces such as gravity. Boundary conditions upon $\mathbf{u}$ and $\mathbf{B}$ depend on the particular problem. If there is no influx of energy from the outside, both kinetic $(\|\mathbf{u}\|_2^2 = \int_\Omega |\mathbf{u}|^2 \, d\sigma)$ and magnetic energy (changing $\mathbf{u}$ to $\mathbf{B}$ in the previous formula) remain bounded for all time. This always happens for standard homogeneous boundary conditions.

For results that do not depend on the particular form of the velocity, the use of the induction equation alone is justified and simpler, since it is linear in $\mathbf{B}$. We will be as general as possible and will not specify boundary conditions for $\mathbf{B}$, except to demand that

$$\frac{1}{T} \int_0^T \int_{\partial\Omega} |\mathbf{B}| \, ds \, dt \leqslant M_1, \tag{2}$$

for some constant M_1 uniform for all time T, which is reasonable enough. Our second assumption is also true for standard boundary conditions of the MHD system: in dimension 2 the mapping $t \rightarrow (\mathbf{u}(t), \mathbf{B}(t))$ is continuous and bounded in time for the Sobolev norm $H^1(\Omega)$ (see, e.g., [7]). We will use only the fact that the sum of kinetic and magnetic energies plus the mean density current

$$\int_\Omega |J| \, d\sigma = \int_\Omega \left| \frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} \right| d\sigma \leqslant M_2$$

is bounded for all time. (Of course the boundedness of $\|\nabla \mathbf{B}\|_2$ in time implies this.)

Now, any solenoidal two-dimensional vector field in a simply connected domain is the curl of a scalar vector field A, i.e., $\partial A/\partial y = B_1$, $\partial A/\partial x = -B_2$. All such possible A's differ by a constant (depending on time). A satisfies the "uncurled" equation

$$(\partial A/\partial t) + \mathbf{u} \cdot \nabla A = \eta \Delta A + C(t) \tag{3}$$

and by making a trivial gauge transformation, i.e., taking instead of A the potential $A-D(t)$, with $D'(t)=C(t)$, one obtains with obvious notational changes

$$(\partial A/\partial t)+\mathbf{u}\cdot\nabla A=\eta\Delta A,\quad A(0)=A_0. \tag{4}$$

This simplification is, however, not free of charge. While we could chose, say, $A(t,\mathbf{x}_0)$ arbitrarily, after the transformation we are totally ignorant of the values of A at any given point. This is relevant because with some hypothesis upon $\mathbf{B}|_{\partial\Omega}$ (such as $\int_{\partial\Omega}|\mathbf{B}|ds$ bounded for all time), by taking $\mathbf{x}_0 \in\partial\Omega$, $A(t,\mathbf{x}_0)=0$, we could get A bounded in $\partial\Omega$ for all time. If, moreover, A satisfies the scalar parabolic equation (4), which as such is subject to the maximum and minimum principles [extrema of A are found at $(\Omega\times\{0\})\cup(\partial\Omega\times[0,\infty))$], A would remain bounded for all time, which would be very convenient. Unfortunately, to demand both things at the same time is an overdetermination. Thus we are bound to allow for indefinite growth of A. However,

$$\begin{aligned}\int_\Omega|A(T)-A_0|d\sigma&\leqslant\int_0^T\int_\Omega\left|\frac{\partial A}{\partial t}\right|d\sigma dt\\&\leqslant\int_0^T\int_\Omega|\mathbf{u}||\nabla A|+\eta|\Delta A|d\sigma dt\\&\leqslant\int_0^T\left(\|\mathbf{u}\|_2\|\mathbf{B}\|_2+\eta\int_\Omega|J|d\sigma\right)dt\leqslant M_3T,\end{aligned} \tag{5}$$

because of our previous statements upon energies and current density. Thus $\|A\|_1$ grows at most linearly in time. Also, by standard energy inequalities,

$$\begin{aligned}\tfrac{1}{2}\int_\Omega A(T)^2d\sigma=\tfrac{1}{2}\int_\Omega A_0^2d\sigma-\eta\int_0^T\int_{\partial\Omega}A\mathbf{B}\cdot d\mathbf{r}\,dt\\-\eta\int_0^T\int_\Omega|\mathbf{B}|^2d\sigma dt.\end{aligned} \tag{6}$$

(Notice that $\partial A/\partial n$ is the tangential component of $\mathbf{B}$.) Thus if the boundary term is bounded or negative (e.g., if the tangential component of $\mathbf{B}$ vanishes), then $\|A\|_2$, and therefore $\|A\|_1$, remain bounded in time.

Finally, notice that the level curves of A, $A=$const are precisely the field lines of $\mathbf{B}$.

II. GLOBAL ESTIMATES

Let $\Phi:(-\infty,\infty)\to[0,\infty)$ be a positive, increasing, and smooth function. Obviously

$$(\partial/\partial t)(\Phi\circ A)=\Phi'(A)(\partial A/\partial t),$$

$$\mathbf{u}\cdot\nabla(\Phi\circ A)=\Phi'(A)(\mathbf{u}\cdot\nabla A),$$

$$\Delta(\Phi\circ A)=\Phi'(A)\Delta A+\Phi''(A)|\nabla A|^2.$$

Therefore the function $\Phi\circ A$ satisfies the equation

$$\frac{\partial}{\partial t}(\Phi\circ A)+\mathbf{u}\cdot\nabla(\Phi\circ A)-\eta\Delta(\Phi\circ A)+\eta\Phi''(A)|\nabla A|^2=0. \tag{7}$$

If we integrate every term in Ω, the second one vanishes:

$$\begin{aligned}\int_\Omega\mathbf{u}\cdot\nabla(\Phi\circ A)d\sigma&=\int_\Omega\nabla\cdot[\mathbf{u}(\Phi\circ A)]d\sigma\\&=\int_{\partial\Omega}(\Phi\circ A)\mathbf{u}\cdot\mathbf{n}\,ds=0,\end{aligned}$$

while the third one equals

$$\begin{aligned}\int_\Omega\Delta(\Phi\circ A)d\sigma&=\int_{\partial\Omega}\frac{\partial(\Phi\circ A)}{\partial n}ds=\int_{\partial\Omega}\Phi'(A)\frac{\partial A}{\partial n}ds\\&=-\int_{\partial\Omega}\Phi'(A)\mathbf{B}\cdot d\mathbf{r}.\end{aligned}$$

Hence, after integrating in time in $[0,T]$,

$$\begin{aligned}\eta\int_0^T\int_\Omega\Phi''(A)|\nabla A|^2d\sigma dt=\eta\int_0^T\int_{\partial\Omega}\Phi'(A)\mathbf{B}\cdot d\mathbf{r}dt\\+\int_\Omega\Phi(A_0)-\Phi(A(T))d\sigma.\end{aligned} \tag{8}$$

Let us choose a specific Φ. For any smooth, compactly, supported real function ϕ, let

$$\Phi(s)=\int_{-\infty}^s(s-v)\phi(v)^2dv.$$

Notice that Φ is the convolution of the identical function in $[0,\infty)$ and ϕ^2. By elementary calculations,

$$0\leqslant\Phi'(s)=\int_{-\infty}^s\phi(v)^2\,dv\leqslant\|\phi\|_2^2, \tag{9}$$

which, by the mean value theorem, implies

$$|\Phi(s_1)-\Phi(s_2)|\leqslant\|\phi\|_2^2|s_1-s_2|. \tag{10}$$

Also

$$\Phi''(s)=\phi(s)^2. \tag{11}$$

Let us bound all the terms in the previous inequality:

$$\begin{aligned}\eta\left|\int_0^T\int_{\partial\Omega}\Phi'(A)\mathbf{B}\cdot d\mathbf{r}dt\right|&\leqslant\eta\|\phi\|_2^2\int_0^T\int_{\partial\Omega}|\mathbf{B}|ds dt,\\\left|\int_\Omega\Phi(A_0)-\Phi(A(T))d\sigma\right|&\leqslant\|\phi\|_2^2\int_\Omega|A_0-A(T)|d\sigma.\end{aligned} \tag{12}$$

We are left with

$$\begin{aligned}\int_0^T\int_\Omega\phi(A)^2|\nabla A|^2d\sigma dt\leqslant\|\phi\|_2^2\\\times\left(\int_0^T\int_{\partial\Omega}|\mathbf{B}|ds dt+\frac{1}{\eta}\int_\Omega|A_0-A(T)|d\sigma\right).\end{aligned} \tag{13}$$

By Cauchy-Schwarz's inequality,

$$\frac{1}{T}\int_0^T\int_\Omega \phi(A)|\nabla A|d\sigma dt \leq m(\Omega)^{1/2}\left(\frac{1}{T}\int_0^T\int_\Omega \phi(A)^2|\nabla A|^2\, d\sigma dt\right)^{1/2},$$

where $m(\Omega)$ denotes the area of Ω. Thus

$$\frac{1}{T}\int_0^T\int_\Omega \phi(A)|\nabla A|d\sigma dt \leq \|\phi\|_2 m(\Omega)^{1/2}\left(\frac{1}{T}\int_0^T\int_{\partial\Omega}|\mathbf{B}|ds dt + \frac{1}{\eta T}\int_\Omega |A_0 - A(T)|d\sigma\right)^{1/2}. \tag{14}$$

By the bounds on the right-hand quantities stated in the Introduction,

$$\frac{1}{T}\int_0^T\int_\Omega \phi(A)|\nabla A|d\sigma dt \leq \|\phi\|_2 m(\Omega)^{1/2}\left(M_1 + \frac{1}{\eta}M_3\right)^{1/2} = M\|\phi\|_2. \tag{15}$$

Now, it is known [8] that for any smooth enough function such as A, for almost every v the level sets $S_v : A = v$ are smooth manifolds (and therefore disjoint union of curves), and for any continuous function G in Ω

$$\int_\Omega G|\nabla A|d\sigma = \int_{-\infty}^{\infty} dv \int_{S_v} G(s)ds,$$

where ds denotes the arclength differential; this result may be vastly generalized. Notice that it is conceivable for **B** to vanish in an open subset of Ω, and therefore for A to be constant there; i.e., for some v's the set S_v may fail to be a union of smooth curves. Obviously there are no field lines in such a region. However, this only may occur for a set of levels of measure zero.

Taking $G = \phi\circ A$, in S_v the function G equals $\phi(v)$, and therefore

$$\int_\Omega \phi(A)|\nabla A|d\sigma = \int_{-\infty}^{\infty}\phi(v)\Lambda(S_v)dv, \tag{16}$$

where $\Lambda(S_v)$ denotes (for almost every v) the length of S_v. Of course the sets S_v and therefore their length may vary in time. If we denote by $\Lambda_T(v)$ the time mean of their lengths,

$$\Lambda_T(v) = \frac{1}{T}\int_0^T \Lambda(S_v)dv,$$

the previous bound yields

$$\int_{-\infty}^{\infty}\phi(v)\Lambda_T(v)dv \leq M\|\phi\|_2.$$

Since the set of possible ϕ's is dense in $L^2(-\infty,\infty)$, we have proved our main bound,

$$\int_{-\infty}^{\infty}\Lambda_T(v)^2 dv \leq M^2. \tag{17}$$

This is a time-independent bound on field line length. As mentioned previously, we must admit that the integral on the left measures not only field line lengths, but also the size of the field. This occurs because an analogous distribution of lengths within an interval of v's will be multiplied by the length of the interval: thus, the more rapidly A varies for a fixed geometrical configuration of field lines, the larger the integral. Since $|\nabla A| = |\mathbf{B}|$, in a sense the integral measures the length of field lines weighted by $|\mathbf{B}|$.

To study some consequences of the above formula we could see what would happen if A remains bounded and M_1 tends to zero in time. Then M also tends to zero, so the integral of Λ_T^2 tends to zero as $T\to\infty$. Hence the function itself tends to zero for almost every v. (To be rigorous, for every sequence $T_n\to\infty$ there exists a subsequence such that $\Lambda_{T_k}\to 0$ almost everywhere.) However, Λ_T is rather peculiar: if two lines $A = v_1$, $A = v_2$ have a sizable length, all the intervening lines $A = v$, $v_1 < v < v_2$, will have at least an intermediate length, which suggests that the only way for Λ_T to tend to zero almost everywhere is to make its support smaller and smaller, i.e., a single set S_v will tend to fill all Ω. In other words, A tends to a constant and therefore **B** to zero. In fact, dividing by T the energy identity (6), all the terms except perhaps the last one tend to zero with our assumptions, so the last one must tend also. Thus **B** tends to vanish in quadratic mean, as expected.

III. LOCAL ESTIMATES

The previous bounds could in principle allow for field lines to be very twisted in some small region of Ω and compensate for this fact by being rather straight in the remaining part. We will prove that local length is bounded by local parameters, in particular local means of the velocity and magnetic field. We visualize field lines as transported by the flow while diffused by resistivity. It is not clear, however, how far the transport must be taken into account in long-term evolution, so our estimates are not so predictable.

As a first approach, one thinks that the length of a curve within a ball of radius r should be at most of order r, unless it is very intrincate there. But since for small r few of the field lines cut the ball, the mean should behave like $rF(r)$, with $F(r)\to 0$ as $r\to 0$. This rough approximation is rather correct, and F may be bounded with the values of the field and velocity within a concentric ball of radius $2r$.

Let $\mathbf{x}_0\in\Omega$, $\overline{B_{2r}} = \overline{B(\mathbf{x}_0,2r)}\subset\Omega$, and ψ be a positive smooth function whose value is 1 in $B_r = B(\mathbf{x}_0, r)$ and vanishing outside B_{2r}, chosen in such a way that (say) $|\nabla\psi| \leq 3/r$. Let Φ be as before: our test function will now be $\psi(\Phi\circ A)$. By elementary calculations it satisfies the evolution equation

$$[(\partial/\partial t) + \mathbf{u}\cdot\nabla - \eta\Delta][\psi(\Phi\circ A)] + \eta\psi\Phi''(A)|\nabla A|^2 = \Phi(A)(\mathbf{u}\cdot\nabla\psi - \eta\Delta\psi) - 2\eta\Phi'(A)\nabla A\cdot\nabla\psi. \tag{18}$$

Since $\psi(\Phi\circ A)$ vanishes in the neighborhood of $\partial\Omega$, the integrals of $\mathbf{u}\cdot\nabla[\psi(\Phi\circ A)]$ and $\Delta[\psi(\Phi\circ A)]$ vanish. Also,

$$\int_\Omega \Phi(A)\Delta\psi\, d\sigma = -\int_\Omega \Phi'(A)\nabla A\cdot\nabla\psi\, d\sigma,$$

$$\int_\Omega \Phi(A)\mathbf{u}\cdot\nabla\psi\, d\sigma = -\int_\Omega \psi\Phi'(A)\mathbf{u}\cdot\nabla A\, d\sigma.$$

Hence, integrating the whole equation in $\Omega\times[0,T]$ (or, equivalently, in $B_{2r}\times[0,T]$),

$$\eta\int_0^T\int_{B_{2r}}\psi\phi(A)^2|\nabla A|^2 d\sigma dt = \int_{B_{2r}}\psi(\Phi(A_0) - \Phi(A(T)))d\sigma - \int_0^T\int_{B_{2r}}\psi\Phi'(A)\mathbf{u}\cdot\nabla A\, d\sigma dt - \int_0^T\int_{B_{2r}}\Phi'(A)\nabla A\cdot\nabla\psi\, d\sigma dt. \tag{19}$$

The first term on the right-hand side may as before be bounded by

$$\|\phi\|_2^2 T\sup_{[0,T]}(\|\mathbf{u}\|_{2,B_{2r}}\|\mathbf{B}\|_{2,B_{2r}} + \eta\|J\|_{1,B_{2r}}) = \|\phi\|_2^2 M_{1,r}T. \tag{20}$$

The second and third terms admit different estimates, after writing $|\Phi'(A)|\leqslant\|\phi\|_2^2$ and $|\nabla A| = |\mathbf{B}|$. If we denote by $\|\mathbf{B}\|_{\infty,2r}$ the maximum of $|\mathbf{B}|$ in B_{2r}, the sum of the second and third terms may be bounded by

$$\|\phi\|_2^2\left(\sup_{[0,T]}\|\mathbf{B}\|_{\infty,2r}\right)\left(\int_0^T\int_{B_{2r}}|\mathbf{u}| + \eta|\nabla\psi| d\sigma dt\right) \leqslant \|\phi\|_2^2 T\left(\sup_{[0,T]}\|\mathbf{B}\|_{\infty,2r}\right)\left(\frac{1}{T}\int_0^T\int_{B_{2r}}|\mathbf{u}|d\sigma dt + 12\pi\eta r\right) = \|\phi\|_2^2 M_{2,r}T, \tag{21}$$

with an obvious notation. We have used $|\nabla\psi|\leqslant 3/r$, $m(B_{2r}) = 4\pi r^2$. If instead we use L^2 norms, we get a bound like

$$\|\phi\|_2^2\, T\left(\frac{1}{T}\int_0^T\int_{B_{2r}}|\mathbf{B}|^2\, d\sigma dt\right)^{1/2}\left[\left(\frac{1}{T}\int_0^T\int_{B_{2r}}|\mathbf{u}|^2 d\sigma dt\right)^{1/2} + 6\eta\sqrt{\frac{\pi}{T}}\right] = \|\phi\|_2^2 M_{3,r}T. \tag{22}$$

By using as before the inequality of Cauchy-Schwarz,

$$\frac{1}{T}\int_0^T\int_{B_r}\phi(A)|\nabla A|d\sigma dt \leqslant r\sqrt{\frac{\pi}{T}}\left(\frac{1}{T}\int_0^T\int_{B_{2r}}\phi(A)^2|\nabla A|^2 d\sigma dt\right)^{1/2} \leqslant \|\phi\|_2 r\sqrt{\frac{\pi}{\eta}}(M_{1,r}+M_{i,r}),$$

for $r=2,3$. Let $N_{i,r} = \sqrt{\pi/\eta}(M_{1,r}+M_{i,r})$. Define as before

$$\Lambda_{T,r}(v) = \frac{1}{T}\int_0^T\Lambda(S_v\cap B_r)dt.$$

By using the same theorems as in the global case, now in the domain B_r, we are left with the local estimate

$$\int_{-\infty}^{\infty}\Lambda_{T,r}(v)^2 dv \leqslant rN_{i,r}. \tag{23}$$

Since integrals in B_{2r} occur in $N_{i,r}$, it is reasonable that $N_{i,r}$ tends to 0 with r. For some cases one of the possible i's may be more advisable than another. If, for instance, the peak of **B** is large at B_{2r} while its energy there remains moderate, $N_{3,r}$ is better. On the other hand, the factor involving the velocity is definitely smaller for $i=2$, so that a moderate maximum of **B** would make this bound preferable. We see that quiescent regions, where the norms of **u** and/or **B** are small, yield a small mean length, emphasizing the local nature of the effect.

IV. CONCLUSIONS

While modeling shows that in a chaotic ideal plasma magnetic field lines tend to become very complicated and lengthy, it is believed that diffusivity will simplify the geometry through reconnection (and therefore release of magnetic energy). We prove and quantify this belief for two-dimensional plasmas by showing that there exist uniform bounds for the mean length of magnetic field lines, the mean also taking into account the field size. Moreover, these bounds are local as well as global, in the sense that the mean length of field lines within a ball of the domain is bounded by parameters depending only on the local behavior of the velocity and the magnetic field.

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